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Object–Subject Split and Superselection Partial States

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In an accompanying work it was shown that any subset of quantum mechanical observables determines an (up to equivalence) unique partial statistical theory with partial states. The crucial role of a so-called basic (e.g., pointer) observable in determining an object–subject split with a well-defined subject was made clear. In this article the subject subsystem is assumed to be quantum mechanical, but such that the basic observable is a superselection one. This leads to superselection partial states and to a different approach to the split. Advantages and disadvantages of the latter approach are discussed.

1. INTRODUCTION

This article is a follow-up of a previous one (Herbut, 1993) in which the quantum mechanical object–subject split with a well-defined subject was derived. It was done in the form of a hybrid, i.e., half quantum mechanical and half classical discrete, *partial* state. By this the new general concept of a partial state was introduced in the framework of a general statistical theory and specified to the mentioned states.

If we want to forget about the partial-states formalism and have a definition of the object–subject split in standard quantum mechanical language, then we may proceed as follows.

Let us start with the vague idea of a *split* with an *ill-defined subject.* The latter is something undefined belonging to that part of the world which is *not* described by a given quantum state. A *split with a well-defined subject*, on the other hand, can be obtained by displacement (or shift) from one with an ill-defined subject (by shrinking the object) as follows:

(i) Let us envisage a composite quantum system, the state space of which is a tensor-product Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. Further, we assume that

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a *state* ρ_{12} (a statistical operator) is given. We start with the assumption that the *object* (O) is the composite system, and the *ill-defined observer or subject* (S) is somewhere in the rest of the world (to which the state does not apply). Further, we imagine a *cut* (/) (as a sort of a dividing line) between them. Symbolically, we write $O/S \equiv (1+2)/\cdots$ (the dots stand for the ill-defined subject in the rest of the world).

(ii) A so-called *basic observable*

$$B_2^0 \equiv \sum_k b_k^0 Q_2^{(k)} \qquad \left(k \neq k' \Rightarrow b_k^0 \neq b_{k'}^0, \quad \sum_k Q_2^{(k)} = 1\right) \tag{1}$$

for the second subsystem (a Hermitian operator in $\mathscr{H}_2$) is defined such that it has a purely discrete spectrum with characteristic values b_k^0 and with $Q_2^{(k)}$ as the corresponding characteristic events (projectors). (The index k enumerates the entire discrete spectrum of B_2^0.) Further, we assume that the probability p_k of the event $(1 \otimes Q_2^{(k)})$ in the state ρ_{12} is positive:

$$p_k \equiv \mathrm{Tr}_{12}((1 \otimes Q_2^{(k)})\,\rho_{12}) > 0 \tag{2a}$$

We call the corresponding $Q_2^{(k)}$ *the subject events*, and their specification completes the definition of the new (well-defined) *subject*.

(iii) Making use of the subject events $Q_2^{(k)}$ and the composite-system state ρ_{12}, one defines the conditional first-subsystem states

$$\rho_1^{(k)} \equiv p_k^{-1}\, \mathrm{Tr}_2((1 \otimes Q_2^{(k)})\,\rho_{12}) \tag{2b}$$

where Tr_2 is the partial trace over subsystem 2.

Now we are prepared to make a *displacement or shift* of the *cut* toward the object:

$$O/S = (1+2)/\cdots \rightarrow O/S \equiv 1/2$$

with the understanding that the subject-events *occur*, and the conditional states $\rho_1^{(k)}$, given by (2b), become, after the shift, the states of the individual objects (subsystem 1), whereas subsystem 2, together with B_2^0 and $Q_2^{(k)}$, plays the role of a well-defined subject (the rest of the world is suppressed).

The mentioned *hybrid* partial-state formalism is based (Herbut, 1993) on *restriction* to the set of composite-system observables

$$\left\{\left(A_1 \otimes \sum_k b_k Q_2^{(k)}\right): A_1 \text{ any}, b_k \text{ any}\right\} = \{(A_1 \otimes f(B_2^0)): A_1 \text{ any}, f \text{ any}\} \tag{3}$$

Here A_1 is any observable for the first subsystem, b_k are any real numbers, and f: $\{b_k^0: \forall k\} \rightarrow R_1$ is any single-valued map. As was explained, the set

$\{f(B_2^0\}: f \text{ any}\}$ of second-subsystem observables is understood to be the set of *classical variables*, attributing to the well-defined subject (after the shift) a classical (purely discrete) state (probability distribution):

$$\{p_k \equiv \mathrm{Tr}_{12}((1 \otimes Q_2^{(k)})\, \rho_{12}) : \forall k\} \tag{4}$$

For the subject to be well defined it is decisive that a basic observable B_2^0 be given. But the classical state of the subject may possibly not be necessary. Perhaps also the subject can be quantum mechanically understood, but so that the role of the basic observable is kept intact. The present article is devoted to an investigation of this possibility.

In this work we broaden the set of second-subsystem observables allowing for incompatibility, but requiring B_2^0 to be a *superselection observable* with respect to the considered ones. In other words, we now take the following set of composite-system observables [notation as in Herbut (1993)]:

$$V' \equiv \{(A_1 \otimes A_2): A_1 \text{ any}, [A_2, B_2^0] = 0, \text{otherwise } A_2 \text{ arbitrary}\} \tag{5}$$

We call the corresponding partial states *superselection states*. They also determine a split with a well-defined subject, but in a somewhat different way. Precisely how this is done is discussed in the last section (Section 6).

In Section 2 we define the statistical theory of superselection states, in Section 3 we prove that we are dealing with a state-distinguishing partial statistical theory, and in Section 4 we demonstrate that the superselection states are canonical. In Section 5 we show that an obvious different way to define superselection states does not give new results.

2. THE STATISTICAL THEORY OF SUPERSELECTION STATES

Elaborating (5), we point out the fact that $[A_2, B_2^0] = 0$ is equivalent to

$$\forall k: \quad [A_2, Q_2^{(k)}] = 0$$

Thus, the compatibility of A_2 with B_2^0 is equivalent to

$$A_2 = \sum_k Q_2^{(k)} A_2 Q_2^{(k)} \tag{6}$$

Now we define a statistical theory, the states of which will be shown to give partial states [cf. Definition 3 in Herbut (1993)] on confinement to V' defined by (5).

Definition 1. We make the following construction and call it *the theory of superselection states* (justification below):

(i) The set of variables W consists of Hermitian operators of the form

$$w \equiv \{(A_1 \otimes \bar{A}_2^{(k)}) \colon \forall k\} \tag{7a}$$

where A_1 is an arbitrary Hermitian operator in $\mathscr{H}_1$, and $\bar{A}_2^{(k)}$ is an arbitrary Hermitian operator in $Q_2^{(k)}\mathscr{H}_2$, $\forall k$ [cf. (1)]. The bar reminds us of the domain. The value set of w consists of the products:

$$\{\{(aa^{(k)} \colon a \in \text{spec}\, A_1,\, a^{(k)} \in \text{spec}\, \bar{A}_2^{(k)}\} \colon \forall k\} \tag{7b}$$

where "spec" stands for "spectrum." The measurement procedure for w is that of a coincidence measurement of $(A_1 \otimes 1)$ and of $(1 \otimes \sum_k \bar{A}_2^{(k)} Q_2^{(k)})$. The result of this measurement is the ordered pair $(a, a^{(k)})$.

(ii) The set of states S_S is made up of all entities

$$s \equiv \{(p_k, \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}) \colon \forall k\} \tag{8}$$

where $\{p_k \colon \forall k\}$ is an arbitrary classical discrete probability distribution, and for $p_k > 0$, $\bar{\rho}_{12}^{(k)}$ is an arbitrary statistical operator in $\mathscr{H}_1 \otimes (Q_2^{(k)}\mathscr{H}_2)$, $\forall k$ (for $p_k = 0$ it need not be defined). The bar reminds us of its domain. The generalized Kronecker symbol $\delta(p_k > 0)$ is by definition one if $p_k > 0$, otherwise it is zero. If

$$\{s_i \equiv \{(p_k^{(i)}, \delta(p_k^{(i)} > 0)\, \bar{\rho}_{12}^{(k,i)}) \colon \forall k\} \colon i = 1, 2, \ldots, I\} \subset S_S \tag{9a}$$

and

$$\{w_i \colon i = 1, 2, \ldots, I\} \tag{9b}$$

is a finite set of statistical weights, then, by definition,

$$s \equiv \sum_i w_i s_i \equiv \{(p_k, \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}) \colon \forall k\} \tag{10a}$$

where

$$\forall k \colon \quad p_k \equiv \sum_i w_i p_k^{(i)} \tag{10b}$$

and for $\forall k$, $p_k > 0$,

$$\bar{\rho}_{12}^{(k)} \equiv \sum_i (w_i p_k^{(i)}/p_k)\, \bar{\rho}_{12}^{(k,i)} \tag{10c}$$

The set $\{(w_i p_k^{(i)}/p_k) \colon i = 1, 2, \ldots, I\}$ is here one of statistical weights.

The preparation procedures for s equal those of the corresponding states ρ_{12} of the composite system, where by "corresponding" we mean an element from the inverse image of the map of states (see Theorem 2 below).

(iii) The average-value formula is

$$\langle w, s \rangle \equiv \sum_k p_k \operatorname{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)}) \bar{\rho}_{12}^{(k)}) \tag{11}$$

where $\operatorname{Tr}_{12}^{(k)}$ is the trace in $\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)$.

Theorem 1. In Definition 1 a state-distinguishing statistical theory is constructed.

Proof. To begin with, we have to show that definition (10a)–(10c) is such that a convex combination of convex combinations is a convex combination (of the constituents in the latter), i.e., that S is a convex set [see Definition 1 in Herbut (1993)]. Let (10a)–(10c) be given, and besides,

$$s_i = \sum_{j(i)} w_{j(i)} s_{j(i)}, \qquad i = 1, 2, \ldots, I$$

where $\forall i$: $\{w_{j(i)}: j(i) = 1, 2, \ldots, J(i)\}$ is a finite set of statistical weights, and $\forall j(i)$: $s_{j(i)} \in S$. Let for each value $i = 1, 2, \ldots, I$

$$s_{j(i)} \equiv \{(p_k^{(j(i))}, \delta(p_k^{(j(i))} > 0)\, \bar{\rho}_{12}^{(k, j(i))}): \forall k\}, \qquad j(i) = 1, 2, \ldots, J(i)$$

Then the definition of a convex combination (10a)–(10c) (that is under investigation) in application to the above system of convex combinations gives

$$p_k^{(i)} = \sum_{j(i)} w_{j(i)} p_k^{(j(i))}, \qquad i = 1, 2, \ldots, I, \quad \forall k$$

and for $\forall k$, $p_k^{(i)} > 0$,

$$\bar{\rho}_{12}^{(k,i)} = \sum_{j(i)} (w_{j(i)} p_k^{(j(i))}/p_k^{(i)})\, \bar{\rho}_{12}^{(k, j(i))}$$

Composing the first convex combination and the second system of convex combinations, one obtains

$$p_k = \sum_i w_i \sum_{j(i)} w_{j(i)} p_k^{(j(i))} = \sum_i \sum_{j(i)} (w_i w_{j(i)})\, p_k^{(j(i))}$$

$\forall k$, $p_k > 0$:

$$\bar{\rho}_{12}^{(k)} = \sum_i (w_i p_k^{(i)}/p_k) \sum_{j(i)} (w_{j(i)} p_k^{(j(i))}/p_k^{(i)})\, \bar{\rho}_{12}^{(k, j(i))}$$

$$= \sum_i \sum_{j(i)} (w_i w_{j(i)} p_k^{(j(i))}/p_k)\, \bar{\rho}_{12}^{(k, j(i))}$$

Hence, indeed

$$s = \sum_i \sum_{j(i)} (w_i w_{j(i)})\, s_{j(i)}$$

and this proves that S is a convex set.

Next we prove that $\langle w, s\rangle$ defined in (11) is convex linear in the states. We assume (9a) and (9b). Then, in view of (10a)–(10c) we have

$$\begin{aligned}\left\langle w, \sum_i w_i s_i \right\rangle &= \sum_k p_k \operatorname{Tr}_{12}^{(k)}(A_1 \otimes \bar{A}_2^{(k)}) \left(\left(\sum_i w_i p_k^{(i)} \Big/ p_k \right) \bar{\rho}_{12}^{(k,i)} \right) \\ &= \sum_i w_i \sum_k p_k^{(i)} \operatorname{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}^{(k,i)}) \\ &= \sum_i w_i \langle w, s_i \rangle\end{aligned}$$

Finally, to prove that the statistical theory at issue is state distinguishing, we assume that for the state s given by (8) and for $s' \equiv \{(p'_k, \delta(p'_k > 0)\, \bar{\rho}_{12}'^{(k)}) \colon \forall k\}$ we have

$$\langle w, s\rangle = \langle w, s'\rangle, \qquad \forall w \in W$$

[cf. (7a)]. Utilizing (11), this amounts to

$$\sum_k p_k \operatorname{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}^{(k)}) = \sum_k p'_k \operatorname{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}'^{(k)}$$

Let $|\psi\rangle_1 \in \mathscr{H}_1$, and for a fixed k' value $|\phi\rangle_2 \in Q_2^{(k')} \mathscr{H}_2$, both normalized vectors, and let us take $A_1 \equiv |\psi\rangle\langle\psi|$, $\bar{A}_2^{(k)} \equiv \delta_{k,k'}\, |\phi\rangle\langle\phi|$. Thus we convert the last equation into

$$p_{k'} \langle\psi|_1 \,\langle\phi|_2\, \bar{\rho}_{12}^{(k')}\, |\psi\rangle_1\, |\phi\rangle_2 = p'_{k'} \langle\psi|_1\, \langle\phi|_2\, \bar{\rho}_{12}'^{(k')}\, |\psi\rangle_1\, |\phi\rangle_2$$

If we now assume *ab contrario* that $p_{k'} = 0 \neq p'_{k'}$ (or the converse of this), we can, evidently, choose $|\psi\rangle_1$ and $|\phi\rangle_2$ so that we obtain a contradiction from the last equation. If $p_{k'} \neq 0 \neq p'_{k'}$, then, in view of the Lemma in the Appendix, we obtain

$$p_{k'} \bar{\rho}_{12}^{(k')} = p'_{k'} \bar{\rho}_{12}'^{(k')}$$

Taking the trace gives $p_{k'} = p'_{k'}$ and leaves $\bar{\rho}_{12}^{(k')} = \bar{\rho}_{12}'^{(k')}$. Hence, $s = s'$. ■

3. SUPERSELECTION STATES ARE PARTIAL STATES

Theorem 2. The statistical theory of superselection states gives *partial states*. The *map of variables* takes V' given by (5) and (6) onto W (see Definition 1) as follows:

$$A_1 \otimes A_2 = A_1 \otimes \left(\sum_k Q_2^{(k)} A_2 Q_2^{(k)} \right) \Rightarrow \{(A_1, \bar{A}_2^{(k)} \equiv A_2 \restriction_{Q_2^{(k)} \mathscr{H}_2}) : \forall k\} \quad (12)$$

where $A_2 \restriction_{Q_2^{(k)} \mathscr{H}_2}$ is the restriction of A_2 to the invariant subspace $Q_2^{(k)} \mathscr{H}_2$. The *map of states* associates with each statistical operator ρ_{12} in $\mathscr{H}_1 \otimes \mathscr{H}_2$ a superselection state s in the following way:

$$\rho_{12} \Rightarrow s \equiv \{(p_k \equiv \mathrm{Tr}_{12}\, \rho_{12}(1 \otimes Q_2^{(k)}),\ \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}) : \forall k\} \quad (13a)$$

$$\forall k, p_k > 0: \quad \bar{\rho}_{12}^{(k)} \equiv p_k^{-1}((1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)})) \restriction_{\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)} \quad (13b)$$

Proof. The map of variables is obviously a bijection of V' onto W. It is evidently measurement-procedure and value-set preserving. The map of states is a surjection of the convex set of all ρ_{12} onto S. Namely, any

$$s \equiv \{(p_k, \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}) : \forall k\} \in S$$

can be obtained from a (corresponding) state of the composite system, e.g., from

$$\rho_{12} \equiv \sum_k p_k \bar{\rho}_{12}^{(k)} (1 \otimes Q_2^{(k)})$$

as easily seen. The map of states is convex linear: Let $\{w_i : i = 1, 2, \ldots, I\}$ be a set of statistical weights, and $\{\rho_{12}^{(i)} : i = 1, 2, \ldots, I\}$ a set of statistical operators in $\mathscr{H}_1 \otimes \mathscr{H}_2$. The map at issue acts as follows:

$$\rho_{12}^{(i)} \Rightarrow s_i \equiv \{(p_k^{(i)}, \delta(p_k^{(i)} > 0)\, \bar{\rho}_{12}^{(k,i)}) : \forall k\}$$

where $\forall k$

$$p_k^{(i)} \equiv \mathrm{Tr}_{12}\, \rho_{12}^{(i)} (1 \otimes Q_2^{(k)})$$

and for $p_k^{(i)} > 0$, we have

$$\bar{\rho}_{12}^{(k,i)} \equiv (p_k^{(i)})^{-1} \left((1 \otimes Q_2^{(k)})\, \rho_{12}^{(i)} (1 \otimes Q_2^{(k)})\right) \restriction_{\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)}$$

Then

$$s \equiv \sum_i w_i s_i \equiv \{(p_k, \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}) : \forall k\}$$

where

$$p_k \equiv \sum_i w_i p_k^{(i)} = \mathrm{Tr}_{12}\left(\left(\sum_i w_i \rho_{12}^{(i)}\right)(1 \otimes Q_2^{(k)})\right)$$

and for $p_k > 0$, we have

$$\begin{aligned}\bar{\rho}_{12}^{(k)} &\equiv \sum_i (w_i p_k^{(i)}/p_k)\, \bar{\rho}_{12}^{(k,i)} \\ &= p_k^{-1}\left((1 \otimes Q_2^{(k)})\left(\sum_i w_i \rho_{12}^{(i)}\right)(1 \otimes Q_2^{(k)})\right)\Big|_{\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)}\end{aligned}$$

Thus, s is the image of $\rho_{12} \equiv \sum_i w_i \rho_{12}^{(i)}$ as claimed.

Further, the map of states is preparation-procedure preserving because the preparation of s is defined as that of any corresponding ρ_{12}, i.e., as that of any element of its inverse image by the map of states (which is, in general, a set).

Finally, we prove that the average value (11) is invariant under the two maps taken together:

$$\begin{aligned}&\mathrm{Tr}_{12}\left(\left(\sum_k A_1 \otimes Q_2^{(k)} A_2 Q_2^{(k)}\right)\rho_{12}\right) \\ &= \sum_k \mathrm{Tr}_{12}((A_1 \otimes Q_2^{(k)} A_2 Q_2^{(k)})(1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)})) \\ &= \sum_k p_k\, \mathrm{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}^{(k)}) = \langle w, s \rangle \quad \blacksquare\end{aligned}$$

What remains is to relate the superselection states to the hybrid ones. For this we must start with the following inclusion relation between subsets of observables in $\mathscr{H}_1 \otimes \mathscr{H}_2$ [cf. (3), (5), and (6)]:

$$\left\{\left(A_1 \otimes \left(\sum_k b_k Q_2^{(k)}\right)\right): \forall A_1, \forall b_k\right\} \subset \{(A_1 \otimes A_2): \forall A_1, A_2, [A_2, B_2^0] = 0\}$$

Theorem 3. The *hybrid* states s_H are *partial states* of the *superselection* states s with respect to the subset of variables $\{v \equiv \{(A_1, b_k \bar{I}_2^{(k)}): \forall k\}: \forall A_1, \forall b_k\}$ ($\bar{I}_2^{(k)}$ being the identity operator in $Q_2^{(k)} \mathscr{H}_2$) in terms of the *map of variables*:

$$v \equiv \{(A_1 \otimes b_k \bar{I}_2^{(k)}): \forall k\} \Rightarrow w \equiv \{(A_1, b_k): \forall k\}$$

and the *map of states*:

$$S \ni s \equiv \{(p_k, \delta(p_k > 0)\, \bar{\rho}_{12}^{(k)}): \forall k\}$$
$$\Rightarrow s_H \equiv \{(\delta(p_k > 0)\, \rho_1^{(k)} \equiv \delta(p_k > 0)\, \mathrm{Tr}_2^{(k)} \bar{\rho}_{12}^{(k)}, p_k): \forall k\} \in S_H$$

where $\mathrm{Tr}_2^{(k)}$ is the partial trace over subsystem 2 in $\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)$.

Proof. Since we have analogous procedures for evaluating the convex combinations in S and in S_H, the preservation of these by the map of states is easily proved. This map preserves also the preparation procedure because both s and its image s_H "inherit" it from one and the same composite-system state ρ_{12} that corresponds to both s and s_H (as easily seen). Finally, that the average value is preserved by the two maps is seen as follows:

$$\begin{aligned}\langle \{(A_1, b_k \bar{I}_2^{(k)}): \forall k\}, s\rangle &= \sum_k p_k \,\mathrm{Tr}_{12}^{(k)}((A_1 \otimes b_k \bar{I}_2^{(k)})\, \bar{\rho}_{12}^{(k)}) \\ &= \sum_k p_k b_k \,\mathrm{Tr}_1(A_1 \rho_1^{(k)}) \\ &= \langle \{(A_1, b_k): \forall k\}, s_H\rangle \quad \blacksquare\end{aligned}$$

In view of transitivity of the partial order, we have the following two chains of partial states:

$$\rho_1 < s_H < s < \rho_{12} \tag{14a}$$

corresponding to the chain of subsets of observables in $\mathscr{H}_1 \otimes \mathscr{H}_2$ (written in a simplified way):

$$\{A_1 \otimes 1\} \subset \left\{A_1 \otimes \sum_k b_k Q_2^{(k)}\right\} \subset \{(A_1 \otimes A_2): [A_2, B_2^0] = 0\} \subset \{A_{12}\} \tag{14b}$$

and

$$\{p_k: \forall k\} < s_H < s < \rho_{12} \tag{15a}$$

corresponding to the chain of sets of observables

$$\begin{aligned}&\left\{1 \otimes \sum_k b_k Q_2^{(k)}\right\} \subset \left\{A_1 \otimes \sum_k b_k Q_2^{(k)}\right\} \\ &\qquad \subset \{(A_1 \otimes A_2): [A_2, B_2^0] = 0\} \subset \{A_{12}\}\end{aligned} \tag{15b}$$

4. THE SUPERSELECTION STATES ARE CANONICAL

One is more motivated to make use of partial states if they are canonical (i.e., the simplest possible).

Theorem 4. The statistical theory of superselection states is *canonical.*

Proof. We have to prove that the entities

$$\{p_k \equiv \mathrm{Tr}_{12}\, \rho_{12}(1 \otimes Q_2^{(k)}): \forall k\}$$
$$\{\bar{\rho}_{12}^{(k)} \equiv p_k^{-1}((1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)}))\,|_{\mathscr{H}_1 \otimes (Q_2^{(k)} \mathscr{H}_2)}: \forall k,\, p_k > 0\}$$

taken together, are necessary and sufficient for distinguishing the equivalence classes in the set of all quantum mechanical composite-system states $\{\rho_{12}\}$ when the equivalence relation is based on V' given by (5) [cf. Herbut (1993), Definition 7]. Taking an observable $(A_1 \otimes A_2) \in V'$, and having in mind (6), the average value that determines the classes, in view of (13a), (13b) amounts to

$$\begin{aligned} &\mathrm{Tr}_{12}((A_1 \otimes A_2)\, \rho_{12}) \\ &\quad = \sum_k p_k\, \mathrm{Tr}_{12}(A_1 \otimes Q_2^{(k)} A_2 Q_2^{(k)})(p_k^{-1}(1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)})) \\ &\quad = \sum_k p_k\, \mathrm{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}^{(k)}) \end{aligned} \tag{16}$$

If ρ_{12} and ρ'_{12} are mapped (by the map of states) into the same superselection state s (i.e., if they determine the same numbers p_k, $\forall k$, and the same statistical operators $\bar{\rho}_{12}^{(k)}$ for each $p_k > 0$), then the rhs of (16) is the same for ρ_{12} and for ρ'_{12} for all $(A_1 \otimes A_2) \in V'$. Hence, necessarily, $\rho_{12} \sim_{V'} \rho'_{12}$. Contrariwise, let us assume that $\rho_{12} \sim_{V'} \rho'_{12}$. Putting $A_1 \equiv 1$ and $A_2 \equiv Q_2^{(k)}$, this entails $p_k = p'_k$ for $\forall k \in K$. If $p_k = p'_k > 0$, then (16) entails, upon substituting k by k' in it and putting $\forall k'$: $\bar{A}_2^{(k')} \equiv \delta_{k',k} \bar{A}_2^{(k)}$,

$$\mathrm{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}^{(k)}) = \mathrm{Tr}_{12}^{(k)}((A_1 \otimes \bar{A}_2^{(k)})\, \bar{\rho}_{12}'^{(k)})$$

Taking $|\phi\rangle_1 \in \mathscr{H}_1$, $A_1 \equiv |\phi\rangle_1 \langle\phi|_1$, $|\psi\rangle_2 \in Q_2^{(k)} \mathscr{H}_2$, $\bar{A}_2^{(k)} \equiv |\psi\rangle_2 \langle\psi|_2$ (normalized vectors), we arrive at

$$\langle\phi|_1 \langle\psi|_2\, \bar{\rho}_{12}^{(k)}\, |\phi\rangle_1 |\psi\rangle_2 = \langle\phi|_1 \langle\psi|_2\, \bar{\rho}_{12}'^{(k)}\, |\phi\rangle_1 |\psi\rangle_2$$

This finally gives (cf. the Appendix) $\bar{\rho}_{12}^{(k)} = \bar{\rho}_{12}'^{(k)}$, and this is so for $\forall k$. Thus, we have proved that the entities making up a superselection state s are characteristic for the equivalence classes, i.e., for the elements of $\{\rho_{12}\}/\sim_{V'}$. ■

5. ARE THERE OTHER SUPERSELECTION STATES?

Inspecting the chain of subsets of observables (14b), one notices that one can interpolate another set between the last two sets without violation of the underlying idea that the basic observable is a superselection observable in the set. This interpolated set is

$$V'' \equiv \{A_{12}\text{: any } A_{12} \text{ satisfying } [A_{12}, (1 \otimes B_2^0)] = 0\}$$

The commutation relation again amounts to [cf. (6)]

$$A_{12} = \sum_k (1 \otimes Q_2^{(k)})\, A_{12}(1 \otimes Q_2^{(k)}) \tag{17}$$

The question is whether one can get other superselection states when one takes the subset V'' (cf. the Introduction), i.e., if one can interpolate some new states between s and ρ_{12} in (14a) in this way.

Theorem 5. The set of observables V'' determines the *same superselection states* as its subset (5), in which A_{12} are required to be of the form $A_1 \otimes A_2$.

Proof. In view of the fact that the partial statistical theory is essentially determined by the equivalence relation defining the classes of statistical operators [cf. Definition 5, Proposition 1, and Theorem 1 in Herbut (1993)], it is sufficient to prove that

$$\sim_{V'} = \sim_{V''} \tag{18}$$

If $\rho_{12} \sim_{V''} \rho'_{12}$, then $\mathrm{Tr}_{12}(A_{12}\rho_{12}) = \mathrm{Tr}_{12}(A_{12}\rho'_{12})$ for $\forall A_{12} \in V''$. Since $V' \subset V''$ [cf. (5)], we have $\rho_{12} \sim_{V'} \rho'_{12}$. Starting with the latter equivalence relation, we know that ρ_{12} and ρ'_{12} determine the same superselection state s [cf. (13a), (13b)]. On the other hand, owing to (17),

$$\mathrm{Tr}_{12}\, A_{12}\rho_{12} = \sum_k p_k\, \mathrm{Tr}_{12}\, A_{12}(p_k^{-1}(1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)}))$$

Defining

$$\bar{A}_{12}^{(k)} \equiv A_{12}\,|_{\mathscr{H}_1 \otimes (Q_2^{(k)}\mathscr{H}_2)}$$

$$\bar{\rho}_{12}^{(k)} \equiv (p_k^{-1}(1 \otimes Q_2^{(k)})\, \rho_{12}(1 \otimes Q_2^{(k)}))\,|_{\mathscr{H}_1 \otimes (Q_2^{(k)}\mathscr{H}_2)}$$

and writing $\mathrm{Tr}_{12}^{(k)}$ for the trace in $\mathscr{H}_1 \otimes (Q_2^{(k)}\mathscr{H}_2)$, we have

$$\mathrm{Tr}_{12}\, A_{12}\rho_{12} = \sum_k p_k\, \mathrm{Tr}_{12}^{(k)}\, \bar{A}_{12}^{(k)}\bar{\rho}_{12}^{(k)}$$

Thus, the lhs can be expressed in terms of the entities making up s (and those making up A_{12}). Since ρ'_{12} determines the same s, this equals also $\mathrm{Tr}_{12} A_{12}\rho'_{12}$, i.e., we also have $\rho_{12} \sim_{V''} \rho'_{12}$. ■

6. HOW WELL IS THE SPLIT DETERMINED IN TERMS OF SUPERSELECTION STATES?

In standard quantum mechanics the shift of the cut $O/S \equiv 12/\cdots \to O/S \equiv 1/2$ mentioned in the Introduction is of interest only if one of the subject events $Q_2^{(k)}$ *occurs* on the new subject (subsystem 2). We assume that the occurrence takes place in an *ideal way*, i.e., in an *ideal measurement* (Lüders, 1951; Messiah, 1961, p. 333). Then the initial composite-system state ρ_{12} goes over into

$$\rho'_{12} \equiv \sum_k p_k((1 \otimes Q_2^{(k)})\,\rho_{12}(1 \otimes Q_2^{(k)})) = \sum_k p_k(\bar{\rho}_{12}^{(k)}(1 \otimes Q_2^{(k)}))$$

where for each k, $p_k > 0$, $\bar{\rho}_{12}^{(k)}$ is the statistical operator in $\mathscr{H}_1 \otimes (Q_2^{(k)}\mathscr{H}_2)$ appearing in the superselection state s corresponding to ρ_{12} [cf. (13b)]. The above decomposition of the after-the-measurement state ρ'_{12} implies the following decomposition of the state $\rho_1 \equiv \mathrm{Tr}_2\,\rho'_{12} = \mathrm{Tr}_2\,\rho_{12}$ of the first subsystem:

$$\rho_1 = \sum_k p_k \rho_1^{(k)}$$

where $\forall k$, $p_k > 0$, $\rho_1^{(k)} \equiv \mathrm{Tr}_2^{(k)}\,\bar{\rho}_{12}^{(k)} = p_k^{-1}\,\mathrm{Tr}_2(\rho_{12}(1 \otimes Q_2^{(k)}))$ are the first-subsystem states appearing explicitly in the hybrid state, but not in the superselection state. In the latter they are only implied (as the subsystem states) by $\bar{\rho}_{12}^{(k)}$, which appear explicitly. Since this decomposition of ρ_1 actually takes place in the new split $O/S \equiv 1/2$, the lack of explicit availability of the $\rho_1^{(k)}$ might be considered to be a drawback of the superselection states in comparison with the hybrid states. On the other hand, the actual terms in the above decomposition of ρ'_{12} are explicitly available in the superselection states, and, reading the decomposition in the reverse direction, they make up (by mixing) the actually present after-the-measurement state ρ'_{12}. This may be an advantage of the superselection states over the hybrid states, because in the latter the $\bar{\rho}_{12}^{(k)}$ states are not available at all (not even implicitly). Thus, the possible statistical correlations in these states are an advantage of the superselection states.

If for some k value, $Q_2^{(k)}$ is a ray projector, say $Q_2^{(k)} \equiv |\phi\rangle_2^{(k)}\,\langle\phi|_2^{(k)}$, then, as easily seen, necessarily one has

$$\bar{\rho}_{12}^{(k)} = \rho_1^{(k)} \otimes |\phi\rangle_2^{(k)}\,\langle\phi|_2^{(k)}$$

(because a pure state cannot establish nontrivial statistical correlations).

If the *occurrence* of the subject events $Q_2^{(k)}$ takes place in a *nonideal way*, i.e., if the corresponding measurement is of the first kind (or repeatable) and nonideal (which is possible if we have degeneracy, i.e., if $\mathrm{Tr}_2\, Q_2^{(k)} > 1$), or even of the second kind (Busch *et al.*, 1991), then it is unknown how ρ_{12} changes in the measurement, i.e., the statistical operators $\{\bar{\rho}_{12}^{(k)}: \forall k,\, p_k > 0\}$, which are constituents of the superselection states [cf. (13a), (13b)], have no physical meaning. Thus, in this case, which is actually more frequent than ideal measurement, the superselection states show no advantage over the hybrid ones. It is actually interesting and not widely known that in whatever way (in whatever measurement) $Q_2^{(k)}$ occurs, the first-subsystem state $\rho_1^{(k)}$ is the same as when the occurrence is ideal (Herbut and Vujičić, 1976, Section 6B).

If the basic observable appears in the context of a macroscopic second subsystem as in the example of Bohr's complementarity principle for such systems [see the Introduction in Herbut (1993)], then we are dealing with bodies of very many degrees of freedom (or subsystems), and we have a large, as a rule, infinite dimensionality of $(Q_2^{(k)}\mathscr{H}_2)$. Here the statistical correlations inherent in the states $\bar{\rho}_{12}^{(k)}$ may be very nontrivial. Though precious physically, they may be irrelevant for the split itself. Hence, it seems that it is the hybrid states and not the superselection ones that are *better adapted* to the task of determining the split in a canonical way.

We end this discussion by repeating the remark from the companion article: the displacement $O/S \equiv 12/\cdots \to O/S \equiv 1/2$ makes sense physically *only if* one makes a *measurement* of the basic observable, because the subject events $Q_2^{(k)}$ have to *occur*. It is precisely this occurrence that is the great mystery of quantum mechanics.

APPENDIX

Lemma. Let A_{12} and A'_{12} be linear operators in the composite-system Hilbert space $\mathscr{H}_1 \otimes \mathscr{H}_2$. Let, further,

$$\langle\psi|_1 \langle\phi|_2 A_{12} |\psi\rangle_1 |\phi\rangle_2 = \langle\psi|_1 \langle\phi|_2 A'_{12} |\psi\rangle_1 |\phi\rangle_2$$

hold for all normalized vectors $|\psi\rangle_1 \in \mathscr{H}_1$, $|\phi\rangle_2 \in \mathscr{H}_2$. Then

$$A_{12} = A'_{12}$$

Proof. Let $B_{12} \equiv A'_{12} - A_{12}$. By assumption, $\forall\, |\psi\rangle_1 \in \mathscr{H}_1$, $\langle\psi|\psi\rangle_1 = 1$, $\forall\, |\Phi\rangle_2 \in \mathscr{H}_2$, $\langle\Phi \mid \Phi\rangle_2 = 1$:

$$\begin{aligned} 0 &= \langle\psi|_1 \langle\Phi|_2 B_{12} |\psi\rangle_1 |\Phi\rangle_2 \\ &= \mathrm{Tr}_{12}(|\psi\rangle_1 |\Phi\rangle_2 \langle\psi|_1 \langle\Phi|_2)\, B_{12} \\ &= \mathrm{Tr}_{12}(|\psi\rangle_1 \langle\psi|_1)(|\Phi\rangle_2 \langle\Phi|_2)\, B_{12} \end{aligned} \tag{A1}$$

Let k and l be two index values in a given orthonormal basis $\{|k\rangle: k=1, 2, \ldots\}$ in $\mathscr{H}_1$ or $\mathscr{H}_2$, and let $k<1$. We define the auxiliary normalized (but nonorthogonal) vectors

$$|a\rangle \equiv 2^{-1/2}\,|k\rangle + 2^{-1/2}\,|l\rangle, \qquad |b\rangle \equiv 2^{-1/2}\,|k\rangle + i2^{-1/2}\,|l\rangle \tag{A2}$$

Straightforward evaluation shows that

$$i\,|a\rangle\langle a| + |b\rangle\langle b| = 2^{-1}(1+i)\,|k\rangle\langle k| + i\,|l\rangle\langle k| + 2^{-1}(1+i)\,|l\rangle\langle l|$$

implying

$$|l\rangle\langle k| = |a\rangle\langle a| - i\,|b\rangle\langle b| + 2^{-1}(i-1)\,|k\rangle\langle k| + 2^{-1}(i-1)\,|l\rangle\langle l| \tag{A3}$$

and by adjoining of this, one obtains

$$|k\rangle\langle l| = |a\rangle\langle a| + i\,|b\rangle\langle b| - 2^{-1}(1+i)\,|k\rangle\langle k| - 2^{-1}(1+i)\,|l\rangle\langle l| \tag{A4}$$

Let $\{|i\rangle_1: i=1, 2, \ldots\} \subset \mathscr{H}_1$ and $\{|p\rangle_2: p=1, 2, \ldots\} \subset \mathscr{H}_2$ be complete orthonormal bases. The general matrix element of B_{12} is

$$\langle i|_1\,\langle p|_2\,B_{12}\,|j\rangle_1\,|q\rangle_2 = \mathrm{Tr}_{12}(|j\rangle_1\,\langle i|_1)(|q\rangle_2\,\langle p|_2)\,B_{12} \tag{A5}$$

Replacing the possible "off-diagonal" operators in $\mathscr{H}_1$ and/or $\mathscr{H}_2$ in the brackets on the rhs of (A5) by a linear combination of "diagonal" ones [utilizing (A3) or (A4) as the case may be], we find that (A1) implies that

$$\langle i|_1\,\langle p|_2\,B_{12}\,|j\rangle_1\,|q\rangle_2 = 0 \quad \blacksquare$$

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